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Non-analyticity of the Callan–Symanzik β-function of two-dimensional O(N) models

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Abstract. We discuss the analytic properties of the Callan–Symanzik β-function $\beta(g)$ associated with the zero-momentum four-point coupling g in the two-dimensional ϕ^4 model with O(N) symmetry. Using renormalization-group arguments, we derive the asymptotic behaviour of $\beta(g)$ at the fixed point g^*. We argue that $\beta'(g) = \beta'(g^*) + O(|g - g^*|^{1/7})$ for $N = 1$ and $\beta'(g) = \beta'(g^*) + O(1/\log|g - g^*|)$ for $N \geqslant 3$. Our claim is supported by an explicit calculation in the Ising lattice model and by a $1/N$ calculation for the two-dimensional ϕ^4 theory. We discuss how these non-analytic corrections may give rise to a slow convergence of the perturbative expansion in powers of g.

1. Introduction

Renormalization-group theory is a very important tool for the understanding of the critical behaviour of statistical models in the neighbourhood of the critical point. We consider models with an N-vector real order parameter and O(N) symmetry. Because of universality, quantitative predictions can be obtained by studying any theory belonging to the same universality class. For the models we are dealing with here, we may consider the Ginzburg–Landau Hamiltonian

$$\mathcal{H} = \int d^d x \left[\frac{1}{2} \left(\partial_\mu \vec{\phi} \right)^2 + \frac{1}{2} r \vec{\phi}^2 + \frac{1}{4!} g_0 (\vec{\phi}^2)^2 \right] \tag{1}$$

where $\vec{\phi}$ is an N-component real field. This Hamiltonian describes many interesting systems at criticality. The liquid–vapour transition in fluids and the infinite-length properties of polymers in dilute solutions correspond to the $N = 1$ (Ising) and $N = 0$ model, respectively; the ^{4}He superfluid phase transition is in the same universality class as the three-dimensional two-component theory (XY model), while the Hamiltonian (1) with $N = 3$ describes isotropic ferromagnetic materials. Three-dimensional N-vector systems and two-dimensional systems with $N < 2$ have a conventional critical behaviour: thermodynamic quantities have power-law

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singularities near the critical point. On the other hand, in two dimensions the XY model shows a Kosterlitz–Thouless transition, while for $N \geqslant 3$ no finite-temperature transition exists: the correlation length diverges only for $T \to 0$. For $N \geqslant 3$ the theory is asymptotically free with a critical behaviour described by the perturbative renormalization group applied to the nonlinear σ-model.

Precise estimates of the critical parameters in the symmetric phase can be obtained using several different methods. One of them, which in many cases provides very precise results, relies on a perturbative expansion in powers of the zero-momentum four-point renormalized coupling g performed at fixed dimension d [1]. The theory is renormalized by introducing a set of zero-momentum conditions for the (one-particle irreducible) two- and four-point correlation functions:

$$\Gamma^{(2)}(p)_{\alpha\beta} = \delta_{\alpha\beta} Z_G^{-1} \left[m^2 + p^2 + \mathrm{O}(p^4)\right] \tag{2}$$

$$\Gamma^{(4)}(0,0,0,0)_{\alpha\beta\gamma\delta} = Z_G^{-2} m^{4-d} g \tfrac{1}{3} \left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\right). \tag{3}$$

For $m \to 0$, the coupling g is driven toward an infrared-stable zero g^* of the corresponding Callan–Symanzik β-function

$$\beta(g) \equiv m \frac{\partial g}{\partial m}. \tag{4}$$

The derivative of the β-function at g^*, $\beta'(g^*)$, is related to the leading non-analytic correction-to-scaling exponent. Usually—but we shall argue here that this may not always be the case—the leading non-analytic corrections are determined by the critical dimension ω_1 of the leading irrelevant operator: in this case, we have $\beta'(g^*) = \omega_1$. At present, $\beta(g)$ has been computed to six loops in three dimensions [2] and to five loops in two dimensions [3].

Perturbative expansions in powers of g are asymptotic. In order to obtain estimates of universal critical quantities, it is essential to resum the perturbative series. This can be done by exploiting their Borel summability and the knowledge of their large-order behaviour (see, e.g., [4] and references therein). The large-order behaviour of the series $S(g) = \sum s_k g^k$ is related to the singularity g_b of the Borel transform $B(g)$ that is closest to the origin. For large k,

$$s_k \sim k!(-a)^k k^b \left[1 + \mathrm{O}(k^{-1})\right] \qquad \text{with} \quad a = -1/g_b. \tag{5}$$

The value of g_b can be obtained by means of a steepest-descent calculation [5, 6]. It depends only on the Hamiltonian, while the exponent b depends on which Green function is considered. If the perturbative expansion is Borel summable, then g_b is negative. Since the Borel transform is singular for $g = g_b$, its expansion in powers of g converges only for $|g| < |g_b|$. An analytic extension can be obtained by a conformal mapping [7], such as

$$y(g) = \frac{\sqrt{1 - g/g_b} - 1}{\sqrt{1 - g/g_b} + 1}. \tag{6}$$

The Borel transform becomes an expansion in powers of $y(g)$ that converges for all positive values of g, provided that all singularities of the Borel transform are on the real negative axis [7]. Therefore, the use of the Borel transform and of the conformal mapping (6) transforms the original asymptotic series into a convergent expansion. Any universal quantity, such as the critical exponents, is estimated by resumming the corresponding perturbative series and by evaluating the resummed function of g at the fixed-point value g^*.

The critical value g^* of the renormalized coupling is a universal quantity. Therefore, it can also be obtained by considering any statistical (lattice) model belonging to the corresponding universality class. Then

$$g^* = \lim_{t\to 0} g(t) \equiv \lim_{t\to 0}\left[-\frac{3N}{N+2}\frac{\chi_4}{\chi^2\xi^d}\right] \tag{7}$$

where $t \equiv T/T_c - 1$, χ is the magnetic susceptibility, ξ the second-moment correlation length and χ_4 the zero-momentum four-point connected correlation function. Using equation (7), one can obtain an independent estimate of g^*.

An important issue in the field-theoretical (FT) approach concerns the analytic properties of $\beta(g)$. General renormalization-group arguments [1, 8, 9] (see also [10, 11]) and explicit calculations to next-to-leading order within the framework of the $1/N$ expansion [12, 13] show that $\beta(g)$ is not analytic at $g = g^*$. This fact may cause a slow convergence of the resummations of the perturbative series to the correct fixed-point value. The reason is that this resummation method approximates the β-function in the interval $[0, g^*]$ with a sum of analytic functions. Since, for $g = g^*$, the β-function is not analytic, the convergence at the endpoint of the interval is slow. This may also lead to an underestimate of the uncertainty that is usually derived from stability criteria. In spite of these problems, in three dimensions, FT results are in good agreement† with the estimates obtained in other approaches [12, 14–19], showing that the above-mentioned non-analyticity causes only very small effects that are negligible in most cases. Using general renormalization-group arguments, for three-dimensional models one expects [8]

$$\beta(g) = -\beta'(g^*)(g^* - g)\left[1 + a_1(g^* - g)^p + a_2(g^* - g) + \cdots\right] \tag{8}$$

where $\beta'(g^*) = \omega_1$ and p is a non-integer exponent that is equal to the smallest of the following exponent combinations: $p = \omega_2/\omega_1 - 1$, where ω_2 is the scaling dimension of the next-to-leading irrelevant operator, $p = 1/\Delta$, where $\Delta = \omega_1\nu$ and $p = \gamma/\Delta - 1$. Note the last exponent that was neglected in [8, 12] and that is due to a subleading correction in $g(t)$ is proportional to t^γ. Such a term is related to the presence of an analytic background in the free energy. For small values of N, we have $\Delta \equiv \omega_1\nu \approx \frac{1}{2}$, $\omega_2/\omega_1 \approx 2$ [20], and $\gamma/\Delta > 2$, so that $p = \omega_2/\omega_1 - 1 \approx 1$. In this case the leading non-analytic term is practically indistinguishable from the analytic one, and therefore, one expects only small systematic deviations. For increasing values of N, p decreases, but at the same time $a_1 \to 0$. Thus, also in this case we expect the non-analytic terms to give rise to small systematic deviations.

The situation worsens in the two-dimensional case which we consider here. As a matter of fact, at variance with the three-dimensional case, two-dimensional FT estimates are much more imprecise [3]. We shall argue here that the large observed deviations are caused by the non-analyticity of the renormalization-group functions at g^*. In order to support this argument, we shall compute the behaviour of the β-function for $g \to g^*$ in two cases in which exact results can be obtained by exploiting different techniques.

First, we shall address the $N = 1$ case (i.e. the Ising universality class) in which conformal field theory (CFT) techniques allow the determination of the whole spectrum of relevant and

† Small discrepancies are only observed for $N = 0$ and 1. For instance, we may compare the estimates of g^* and ω_1 obtained using the fixed-dimension FT approach with the apparently best estimates obtained from the analysis of high-temperature (HT) expansions and from Monte Carlo (MC) simulations for lattice models in the same universality class. For $N = 1$, the analysis of the fixed-dimension FT expansion gives $g^* = 1.411(4)$ and $\omega_1 = 0.799(11)$ [14], to be compared with the lattice results $g^* = 1.402(2)$ [15] (HT) and $\omega_1 = 0.845(10)$ [16] (MC). The results are in better agreement for $N = 2$: the analysis of the fixed-dimension g-expansion leads to $g^* = 1.403(3)$ and $\omega_1 = 0.789(11)$ [14], to be compared with $g^* = 1.396(4)$ [17] (HT) and $\omega_1 = 0.79(2)$ [18] (MC).

irrelevant operators of the theory. We shall first show that CFT predicts $\omega_1 = 2$ for the renormalization-group dimension of the leading irrelevant operator, excluding $\omega_1 = \frac{4}{3}$, as has sometimes been claimed. Then, we will consider the lattice Ising model and we will show that equation (8) holds with $\beta'(g^*) = \gamma/\nu = \frac{7}{4}$ and $p = \frac{1}{7}$. Note that in this case $\beta'(g^*) \neq \omega_1 = 2$†. We will then argue that this is the generic behaviour one should expect for models in the Ising universality class. At variance with the three-dimensional case, here p is very small and thus it may be responsible for large systematic deviations in the resummation of the perturbative series. In appendix B we study some simple Borel-summable asymptotic series behaving as (8) with $p = \frac{1}{7}$. We apply the resummation method described above, finding very poor estimates of $\beta'(g^*)$ with largely underestimated error bars.

Second, we shall study the multicomponent ϕ^4 theory with $N \geqslant 3$. Since the model is asymptotically free, we can predict $\omega_1 = 2$ and we can show that logarithmic corrections should be expected at the critical point. A large-N calculation confirms the theoretical predictions.

2. $N = 1$, ϕ^4 theory in $d = 2$

Let us first consider the Ising case, i.e. the case in which the field $\phi(x)$ in the ϕ^4 Hamiltonian is a one-component real field. In [7] the four-loop series of $\beta(g)$ is analysed using the resummation procedure presented in the introduction: they obtain $g^* = 15.5(8)$ and $\beta'(g^*) = 1.3(2)$. Reference [3] computes the five-loop contribution and presents an analysis of the extended series using a Padé–Borel resummation: they obtain‡ $g^* = 15.39(25)$ and $\beta'(g^*) = 1.31(3)$. These results for g^* do not agree with the very precise estimates obtained by a transfer-matrix analysis of the standard square-lattice Ising model [21], $g^* = 14.697\,35(3)$, and by exploiting the form-factor bootstrap approach [22], $g^* = 14.6975(1)$ (see also [12, 23, 24] for high-temperature results). The result for $\beta'(g^*)$ has been interpreted [3, 4] as an indication in favour of the exact result $\beta'(g^*) = \frac{4}{3}$ that would imply the existence of an irrelevant operator with $\omega_1 = \frac{4}{3}$. However, the corresponding scaling corrections do not appear in the standard lattice Ising model in which, thanks to the known exact results (see, e.g., [25–27]), a detailed analysis of the leading correction terms is possible. In principle, this fact does not imply that the interpretation of [3, 4] is wrong, since it could be simply explained by the absence of the corresponding irrelevant operator in the lattice Ising model, which is only one of the possible realizations of the ϕ^4 universality class. However, we shall show below that this is not the case and that no subleading operator with $\omega_1 = \frac{4}{3}$ exists in any unitary model belonging to the Ising universality class. In particular, it does not exist in the $N = 1$, ϕ^4 theory.

Let us briefly comment on this last point. The $\omega_1 = \frac{4}{3}$ interpretation was supported by the fact that an operator with renormalization-group dimension $\omega_1 = \frac{4}{3}$ exists in a particular non-unitary extension of the Ising universality class which is conjectured to describe Ising percolation. However, such an operator can only exist in non-unitary theories, and consequently, it cannot be observed in the unitary ϕ^4 theory. We shall argue in this paper that the estimate of ω_1 obtained within the framework of the perturbative expansion at fixed dimension is strongly affected by non-analytic corrections in the β-function. The fact that one obtains $\beta'(g^*) \simeq \frac{4}{3}$ is only a coincidence and is not related to the presence of the non-

† This is due to the fact that the correction-to-scaling term with the smallest exponent appearing in $g(t)$ is t^γ and not $t^{\omega_1\nu}$. A correction term proportional to t^γ in $g(t)$ is due to the presence of an analytic background in the free energy.

‡ We applied the Le Guillou–Zinn–Justin resummation method [4, 7], using the conformal mapping (6), to the five-loop series of [3]: we obtained substantially equivalent results.

unitary operator with $\omega_1 = \frac{4}{3}$ mentioned above. In order to clarify the issue we have added in appendix A a discussion on the non-unitary extension of the Ising universality class and its relation with the Ising percolation problem.

The only ingredients that are needed to extend the Ising result—the absence of an exponent $\omega_1 = \frac{4}{3}$—to the most general unitary model in the $N = 1$ ϕ^4 universality class are Wilson's renormalization group and some basic results of CFT.

In Wilson's approach, we can rewrite $\mathcal{H}$ as

$$\mathcal{H} = \mathcal{H}^* + \sum_{\{\mathcal{O}\}} u_{\mathcal{O}}(m)\mathcal{O} \tag{9}$$

where $\mathcal{H}^*$ is the fixed-point Hamiltonian, $\{\mathcal{O}\}$ is a complete set of operators and $u_{\mathcal{O}}(m)$ is the corresponding nonlinear scaling fields depending on the inverse correlation length m. Then, we observe that the ϕ^4 theory is unitary. This can be proved to all orders of perturbation theory. It can also be proved non-perturbatively by considering the lattice regularization of the model (1). Indeed, the lattice theory corresponding to (1)—and, of course, also the standard Ising model which is a particular limit of the lattice ϕ^4 theory—with nearest-neighbour couplings is exactly reflection positive, a property that guarantees the unitarity of the Minkowski theory. At the critical point the theory becomes conformally invariant. Now the main point is that within the framework of CFT there exists a complete classification of all possible Z_2 symmetric unitary theories [28, 29]. Moreover, their operator content is exactly known. This means that all dimensions of the operators $\mathcal{O}$ that may appear in equation (9) giving rise to a unitary theory are known exactly†. In particular, no operator with dimension $\omega_1 = \frac{4}{3}$ exists.

According to the CFT analysis [21, 31], the leading irrelevant operator is $T\bar{T}$, where T denotes the energy–momentum tensor, which is expected to give rise to corrections of the order of t^2, where t is the reduced temperature. On the square lattice—but not in a rotationally invariant model or on lattices with different rotational symmetry, for instance, on the triangular lattice—one must also consider a second operator, $\mathcal{T} = T^2 + \bar{T}^2$, which is degenerate with the first one. While $T\bar{T}$ is rotationally invariant, $\mathcal{T}$ breaks rotational invariance and has only the reduced symmetry of the square lattice. Since the correlation function of $\mathcal{T}$ with rotationally invariant operators vanishes, such an operator should not contribute at order t^2 to observables that are rotationally invariant, but only at order t^4 (indeed, $\langle \mathcal{T}_x \mathcal{T}_y \mathcal{O}\rangle$ does not vanish even if $\mathcal{O}$ is rotationally invariant). Of course, $\mathcal{T}$ should contribute to order t^2 to observables that have an angular dependence (an explicit example will be given below).

In recent years there has been extensive work trying to understand the origin of the subleading corrections in the lattice Ising model. The unexpected result is the fact that no correction-to-scaling term due to $T\bar{T}$ has been observed. Let us review the evidence for this fact.

(a) The analysis of the susceptibility [32–34] for $h = 0$ indicates that the corrections of the order of t, t^2, t^3 can be interpreted as purely analytic ones.
(b) The analysis of the free energy on the critical isotherm as a function of h [31] does not find any evidence of correction-to-scaling terms that can be associated with $T\bar{T}$.
(c) The analysis of the free energy, correlation length and susceptibility at the critical point in a finite box [35, 36] shows the presence of corrections with $\omega_1 = 2$. These corrections, however, appear to be due to $\mathcal{T}$ only. Indeed, they are not present on the triangular and

† Let us stress that our argument is by no means original. It was, for instance, already present in [30] that appeared right after the classification of unitary CFTs. In this respect, our main new contribution is the exact calculation of p and the use of this result (discussed in detail in appendix B) to show the relevance of non-analytic corrections in the FT Callan–Symanzik β-function.

honeycomb lattices [35]—on these lattices $\mathcal{T}$ cannot contribute and the first expected correction has $\omega_1 = 4$—and moreover, the dependence of these corrections on the shape of the box is consistent with the behaviour expected for a spin-four operator as $\mathcal{T}$ is [36].

Here, we want to add further evidence for the absence of $T\bar{T}$ by considering the observables characterizing the large-distance behaviour of the two-point function on a square lattice. Indeed, for $x \to \infty$ we can write [37]

$$\langle \sigma_0 \sigma_x \rangle = Z(\beta) \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{\mathrm{e}^{\mathrm{i}p\cdot x}}{\hat{p}^2 + M(\beta)^2} \tag{10}$$

where $\hat{p}^2 = 4\sum_\mu \sin^2(p_\mu/2)$ and the integration is extended over the first Brillouin zone. The quantities $Z(\beta)$ and $M(\beta)$ are known exactly [37]. For $t \equiv 1 - \beta/\beta_c \to 0$, we can write

$$Z(\beta) = \left(128\sqrt{2}\beta_c\right)^{1/4} u_t^{1/4} v_h^2 \left[1 + \mathrm{O}(u_t^4)\right] \tag{11}$$

$$M(\beta)^2 = 16\beta_c^2 u_t^2 \left[1 + \beta_c^2 u_t^2 + \mathrm{O}(u_t^4)\right] \tag{12}$$

where u_t is the nonlinear scaling field associated with the reduced temperature at zero magnetic field h and v_h is related to the nonlinear scaling field u_h associated with h by $u_h = h v_h + \mathrm{O}(h^3)$. Explicitly, [21, 33, 38]

$$u_t = t\left(1 + \frac{\beta_c}{\sqrt{2}}t + \frac{7\beta_c^2}{6}t^2 + \frac{17\beta_c^3}{6\sqrt{2}}t^3 + \mathrm{O}(t^4)\right) \tag{13}$$

$$v_h = 1 + \frac{\beta_c}{\sqrt{2}}t + \frac{23\beta_c^2}{16}t^2 + \frac{191\beta_c^3}{48\sqrt{2}}t^3 + \mathrm{O}(t^4). \tag{14}$$

Using (10) we can derive the angle-dependent correlation length $\xi(\theta)$ defined from the large-distance behaviour of the two-point function along a direction forming an angle θ with the side of the lattice. Using the expression of $\xi(\theta)$ in terms of $M(\beta)$ reported, for example, in [39, 40], we obtain†

$$\xi(\theta) = \frac{1}{4\beta_c u_t}\left[1 + \tfrac{1}{6}\beta_c^2 \cos(4\theta) u_t^2 + \mathrm{O}(u_t^4)\right]. \tag{15}$$

Thus, we see analytically that no correction of the order of $\mathrm{O}(t^2)$ appears in the on-shell renormalization constant $Z(\beta)$—both $T\bar{T}$ and $\mathcal{T}$ are absent. In $\xi(\theta)$ a $\mathrm{O}(t^2)$ correction does appear as already observed in [41]. However, it is proportional to $\cos(4\theta)$, and thus it is due only to the leading operator breaking rotational invariance. No contribution from the rotationally invariant operator $T\bar{T}$ appears.

The expansion of the Callan–Symanzik β-function can be derived using the same arguments employed by Nickel [8, 9] in three dimensions. Let us first consider the lattice Ising model and the coupling $g(t)$ defined in (7) as a function of the reduced temperature. The expansion of χ and χ_4 is well established:

$$\chi = C_2 u_t^{-7/4} v_h^2 \left(1 + p_1 u_t^{7/4} + p_2 u_t^{11/4} \log u_t + p_3 u_t^{11/4} + \cdots\right) \tag{16}$$

$$\chi_4 = C_4 u_t^{-11/2} v_h^4 \left(1 + p_4 u_t^{11/4} + \cdots\right) \tag{17}$$

where C_2, C_4, p_1, p_2, p_3 and p_4 are known constants [21, 22, 25, 27, 33, 42]. In particular, $p_1 = -0.108\,1812\ldots$ Next, we determine the asymptotic behaviour of $\mu_2 \equiv \sum_x x^2 \langle \sigma_0 \sigma_x \rangle = 4\chi\xi^2$

† In particular, the correlation lengths along the side ($\theta = 0$) and the diagonal ($\theta = \pi/4$) of the lattice are given, respectively, by $\xi_s^{-1} = -\ln\tanh\beta - 2\beta$ and $\xi_d^{-1} = -\sqrt{2}\ln\sinh 2\beta$.

from its high-temperature (HT) expansion. The analysis of the 52nd-order HT expansion† of μ_2 shows that its Wegner expansion can be written as

$$\mu_2 = A_2 u_t^{-15/4} v_h^2 \left(1 + p_5 u_t^2 + \cdots\right). \tag{18}$$

The constant p_5 has been computed with high accuracy in the following way. We have first defined a new series s obtained by expanding in powers of β the quantity $(\mu_2 u_t^{15/4} v_h^{-2}/A_2 - 1)u_t^{-2}$, where $A_2 = 1.238\,136\,098$, and u_t, v_h are given by equations (13) and (14) truncated at order t^3 included. Then, we analysed s by means of first-order inhomogeneous integral approximants biased to have a singularity at $\beta = \beta_c$. We verified that the critical exponent associated with the singularity is positive and then computed the value of s for $\beta = \beta_c$. We obtain finally‡ $p_5 = -0.388\,720(3)$. It follows that

$$g(t) = g^* \left[1 - p_1 u_t^{7/4} - p_5 u_t^2 + \mathrm{O}(u_t^{11/4} \log u_t)\right]. \tag{19}$$

Since the second-moment mass $m(t) = 1/\xi(t) = (4\chi/\mu_2)^{1/2}$ scales as

$$m(t)^2 = \frac{4C_2}{A_2} u_t^2 \left[1 + \mathrm{O}(u_t^{7/4})\right] \tag{20}$$

we obtain for the square-lattice Ising§ β-function

$$\begin{aligned}\beta(g) \equiv m\frac{\partial g}{\partial m} &= 2m^2 \left(\frac{\mathrm{d}m^2}{\mathrm{d}u_t}\right)^{-1} \frac{\mathrm{d}g}{\mathrm{d}u_t} \\ &= -\tfrac{7}{4}\Delta g \left(1 + b_1|\Delta g|^{1/7} + b_2|\Delta g|^{2/7} + b_3|\Delta g|^{3/7} + \cdots\right)\end{aligned} \tag{21}$$

where $\Delta g \equiv g^* - g$, and for the non-universal constant b_1, $b_1 = p_5(-g^* p_1)^{-1/7}/(7p_1) \approx 0.480(4)$. It follows that $\beta'(g^*) = \frac{7}{4}$ and $p = \frac{1}{7}$. Let us stress again that this value of $\beta'(g^*)$ is not related to the exponent of the leading irrelevant operator that we expect to be two. This phenomenon occurs whenever $\gamma < \omega_1\nu$. Indeed, in $g(t)$ there is a correction-to-scaling term proportional to t^γ because of the presence of an analytic background in the free energy [32]. If $\gamma < \omega_1\nu$, it represents the leading non-analytic correction in $g(t)$ and therefore $\beta'(g^*) = \gamma/\nu \neq \omega_1$. It should be noted that such a phenomenon does not arise in three-dimensional O(N) models, where the leading non-analytic corrections are determined by the leading irrelevant operator. For instance, for the three-dimensional Ising model $\gamma/\nu = 2 - \eta \simeq 1.96 > \omega_1 \simeq 0.8$. We also mention the recent result $\beta'(g^*) \simeq 1.88$ obtained in [46] using a numerical approach based on the high-temperature expansion of the Ising model, which is not too far from our exact prediction $\frac{7}{4}$.

Now, the question is: what behaviour should we expect for the ϕ^4 field theory? In other words, does equation (19) holds for a generic model in the $N = 1$, ϕ^4 universality class or

† The HT expansion of μ_2 can be found to $\mathrm{O}(\beta^{36})$ in [8]. The 52nd-order series has been kindly provided by Tony Guttmann [43].

‡ It is worth noting that $p_5 \approx -\frac{1}{2}C_2/A_2 = -0.388\,722\ldots$ within error bars, so that μ_2 can be written as $\mu_2 = A_2 v_h^2 u_t^{-15/4} - \frac{1}{2}\chi + \cdots$. This equation may be explained in terms of a momentum dependence of the scaling field u_h. Indeed, μ_2 is not a zero-momentum quantity and thus it is related to the free energy in the presence of a non-uniform magnetic field $h(x)$. However, in this case we expect additional contributions to the scaling fields, proportional to derivatives of $h(x)$ [44]. Our result for μ_2 can be explained if the scaling field u_h is a functional of $h(x)$ with a small-momentum behaviour $u_h = u_h|_{h=\mathrm{constant}} - \frac{1}{8}\partial^2 h(x) +$ higher derivatives. Note also that the u_t^2 term in μ_2 cannot be interpreted as a contribution due to irrelevant operators. Indeed, we do not expect $\mathrm{O}(u_t^2)$ contributions associated with $T\bar{T}$, nor with the non-rotationally invariant $\mathcal{T}$, since μ_2 is a rotationally invariant quantity. This point needs further investigation.

§ Note that in the lattice model g approaches g^* from *above* as $t \to 0$, while in the FT model the opposite happens. For a discussion see [45] and references therein.

are some terms absent? And, in particular, are the conditions $p_1 \neq 0$ and $p_2 \neq 0$ a particular feature of the lattice Ising model only? Sometimes, see, for example, [47] and the discussion of [12], it is conjectured that the β-function is analytic in FT models. However, it was shown in [12] that this conjecture is not true: in the large-N limit, non-analytic terms are indeed present. Unfortunately, in the two-dimensional case for $N = 1$, we do not have any analytic control on the corrections to $\beta(g)$. Nonetheless, we conjecture that equation (21) also holds for the FT $N = 1$ model—of course, with different coefficients b_1, b_2 since the β-function is not universal. We have essentially two arguments to support our conjecture.

(a) We do not see any reason why the bulk term that originates the $p_1 t^{7/4}$ contribution in (19) should not be present. Indeed, the analytic contribution is not a lattice artefact but has a well defined FT meaning. In the CFT framework, it can be considered as a signature of the Identity operator and of its conformal family. Thus, also for the FT model, we expect $p_1 \neq 0$.
(b) A t^2 correction is certainly present in $g(t)$, since we expect the operator $T\bar{T}$ to be present in the FT model. Thus p_5 will not be zero in (19), although it will no longer be related to the correction appearing in μ_2.

It is important to note that the strong non-analytic corrections at $g = g^*$ we have found may explain the large observed deviations among the perturbative FT estimates of g^* and $\beta'(g^*)$, the high-precision numerical results for g^*, and our prediction for $\beta'(g^*)$. As a test, in appendix B we have considered a simple Borel-summable function that has an asymptotic behaviour of the form (21). We have applied the standard resummation method presented above, observing large systematic deviations at $g = g^*$ and a systematic underestimate of the error bars. We should note that these discrepancies, although providing support for the presence of strong non-analytic corrections at $g = g^*$, do not support our specific expansion (21). Indeed, even if $p_1 = 0$ in (19), neglecting logarithmic terms, we would obtain

$$\beta(g) = -2\Delta g\left(1 + c_1|\Delta g|^{3/8} + \cdots\right). \tag{22}$$

Thus, also in this case, there would be a strong non-analytic correction.

3. $N \geqslant 3$, ϕ^4 theory in $d = 2$

Let us now consider the multicomponent ϕ^4 theory with $N \geqslant 3$. For $N = 3$, the Padé–Borel analysis of the five-loop series [3] yields the estimates $g^* = 12.00(14)$ and $\beta'(g^*) = 1.33(2)$. The result for g^* is in reasonable agreement with the more precise estimate $g^* = 12.19(3)$ obtained by employing the form-factor bootstrap approach [22, 48]. We shall now argue that the estimate $\beta'(g^*) \approx \frac{4}{3}$ is again incorrect and that the correct value should instead be $\beta'(g^*) = 2$.

The standard scenario predicts that, for $N \geqslant 3$, the theory is massive for all temperatures. The critical behaviour is controlled by the zero-temperature Gaussian point and can be studied in perturbation theory in the corresponding N-vector model. One finds only logarithmic corrections to the purely Gaussian behaviour. It follows that the operators have dimensions that coincide with their naive (engineering) dimensions, apart from logarithmic multiplicative corrections related to the so-called anomalous dimensions. The leading irrelevant operator has dimension two [49] and thus, for $m \to 0$, we expect [50]

$$g(m) = g^*\left\{1 + cm^2\left(-\ln m^2\right)^{\zeta}\left[1 + \mathrm{O}\left(\frac{\ln(-\ln m^2)}{\ln m^2}\right)\right]\right\} \tag{23}$$

where ζ is an exponent related to the anomalous dimension of the leading irrelevant operator and c is a constant. A one-loop calculation within the framework of the O(N) σ model gives $\zeta = 2/(N-2)$ [49]. Differentiating with respect to the mass, one obtains

$$\beta(g) = m\frac{\partial g}{\partial m} = -2\Delta g\left(1 + \frac{\zeta}{\ln \Delta g} + \cdots\right) \tag{24}$$

with $\Delta g \equiv g^* - g$. Therefore, one expects $\beta'(g^*) = 2$ with logarithmic corrections.

The expansion (24) for $\beta(g)$ is confirmed by a next-to-leading order calculation within the framework of the large-N expansion. Indeed, using the expression for $\beta(g)$ reported in [13] and performing an asymptotic expansion around g^* (see appendix C for details), one finds

$$\beta(g) = -2(g^* - g)\left\{1 + \frac{1}{N}\left[\frac{2}{\ln\Theta}\left(1 + \frac{l(\Theta)}{\ln\Theta}\right) + \frac{5}{2\ln^2\Theta} + \mathrm{O}\left(\frac{l(\Theta)^2}{\ln^3\Theta}\right)\right]\right\} \tag{25}$$

where $l(\Theta) \equiv \ln(-2\ln\Theta)$ and $\Theta \equiv (g^* - g)/g^*$. Comparing equation (25) with equation (24) we obtain $\zeta = 2/N + \mathrm{O}(1/N^2)$, in agreement with the above-mentioned result $\zeta = 2/(N-2)$.

Thus, for $N \geqslant 3$ we predict very strong non-analytic corrections at $g = g^*$. A numerical study on a function with the asymptotic behaviour (24) (see appendix B) shows that such corrections give rise to a slow convergence of the perturbative resummations. In particular, the estimate of ω_1 may be incorrect in spite of the stability of the results with the number of loops considered in the analysis. It is thus not surprising that [3] find $\beta'(g^*) \simeq \frac{4}{3}$ instead of the correct result $\beta'(g^*) = 2$.

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Appendix A. Non-unitary extension of the Ising model

The $\frac{4}{3}$ operator appears in a non-unitary extension of the Ising model that describes Ising percolation.

Let us first of all explain what we mean with the notion of 'non-unitary extension' of the Ising universality class. The starting point is the classification of the minimal unitary conformal field theories discussed in [28, 29].

The operator content of the unitary CFTs *that only possess a* Z_2 *symmetry* (such as the ϕ^4 theory and its multicritical generalizations) is defined by the weights

$$h_{p,q} = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)} \tag{A1}$$

with $m = 3, 4, 5, \ldots$ and the constraints $1 \leqslant p \leqslant (m-1)$, $1 \leqslant q \leqslant p$. The relation between h and the renormalization-group eigenvalue y is $y = 2 - 2h$. For the Ising model $m = 3$. Higher values of m correspond to multicritical Ising-like models (i.e. theories with a Z_2 symmetric potential with powers up to ϕ^{2m-2}). These are the continuum-limit CFTs that correspond to the models introduced in [51, 52]. With $m = 3$ we have only three allowed combinations of (p, q): (1, 1), (2, 1) and (2, 2) that correspond to the identity, energy and spin operators of the Ising model. They are called 'primary' fields. From any one of these primary fields one then has an infinite tower of 'secondary' fields whose scaling dimensions are shifted by integers with respect to those of the primary fields. Since in the Ising model *all the primary fields are relevant*, all the irrelevant fields must be shifted by integers, hence they cannot be distinguished from the analytic corrections. This is the only model in which this happens. In all other models with $m > 3$, there are primary fields that are irrelevant and hence are candidates for non-trivial subleading scaling dimensions.

Besides unitary theories, there is an infinite set of non-unitary ones for all the rational (but non-integer) values of m. Apart from the fact that they do not fulfil unitarity, they have the same properties as those with integer m. In particular, their operator content is completely known and closed expressions for the correlators exist. These models (with both integer and non-integer values of m) are usually called rational conformal field theories (RCFT).

However, this is not the end of the story. In the last few years it has been realized that it is also possible to give a meaning, within the framework of the so-called logarithmic conformal field theories (LCFT) [53], to more general theories, obtained by including in the operator algebra some of the operators corresponding to the values of p and q excluded in equation (A1) [54].

For instance, in the Ising case (i.e. $m = 3$) in which we are interested one should enlarge the set of operators of the standard Ising CFT to those of the type $h_{3,n}$, $n = 1, 2, \ldots$ and $h_{k,4}$, with $k = 1, 2 \ldots$. The LCFT obtained in this way is what we mean by a 'non-unitary extension' of the Ising model.

Despite the fact that these theories are much more difficult to study than the standard RCFTs, several interesting results have been obtained in these last few years (for a recent account see, for instance, [55, 56] and references therein). For the purpose of the present paper we only need to know the scaling dimensions of the new operators. These can be easily obtained by looking at equation (A1).

In particular, in the Ising case, we see that $h_{3,1} = \frac{5}{3}$ hence $y_{3,1} = -\frac{4}{3}$, which is exactly the irrelevant operator that we are looking for. Further examples of such operators (only the relevant ones are listed) are:

$$h_{3,2} = \tfrac{35}{48} \qquad \text{hence} \quad y_{3,2} = \tfrac{13}{24}$$
$$h_{3,3} = \tfrac{1}{6} \qquad \text{hence} \quad y_{3,3} = \tfrac{5}{3}$$
$$h_{2,4} = \tfrac{5}{16} \qquad \text{hence} \quad y_{2,4} = \tfrac{11}{8}.$$

Note that, for all the values $m > 3$ (i.e in the multicritical models), equation (A1) admits a unitary, well defined, operator of type $h_{3,1}$ with weight $(m+2)/m$ so that $y = -4/m$. Thus, a naive limit $m \to 3$ would lead to an operator with $y = -\frac{4}{3}$, This argument is usually given to support the existence of a scaling operator with $\omega_1 = \frac{4}{3}$ (see, e.g., [4]). However, as we have seen, exactly for $m = 3$ this operator becomes 'borderline' and it no longer belongs to the Ising universality class, but only to its non-unitary extension. Thus, the limit $m \to 3$ of equation (A1) cannot be considered as an indication in favour of the presence of a $\omega_1 = \frac{4}{3}$ field in the (unitary) Ising universality class, which the ϕ^4 theory belongs to.

Another context in which the $y = \frac{4}{3}$ field appears, which is completely independent and allows us to share some more light into its meaning, is the Coulomb gas approach to the q-state Potts models due to Nienhuis [57]. By mapping the Potts model in a suitable vertex-type model Nienhuis was able to identify both the leading and the subleading thermal and magnetic operators as a function of q. For $q = 2$ the subleading thermal operator is exactly $y = -\frac{4}{3}$ and the subleading magnetic operator is $y = \frac{13}{24}$ (see also [58]). However, as already noted in [57], *these are operators of the vertex model and not of the Ising model* and they decouple for

$q = 2$. In other words, the vertex model of Nienhuis is a good candidate for an exactly solvable model whose continuum limit is the non-unitary extension of the Ising model. If one requires the vertex model to have a 'physical spectrum' according to the definition given in [57], then one selects only the operators of the standard Ising model and the $y = \frac{4}{3}$ operator decouples. The requirement of having a 'physical spectrum' is equivalent to imposing unitarity on the model.

It would be nice to have some kind of insight into the physical meaning of the above-mentioned operators directly from the Ising model. Some hints in this direction are given by the so-called 'Ising percolation' problem, i.e. the behaviour of the Coniglio–Klein clusters in the Ising model. It turns out that the *relevant* operators in the non-unitary extension of the Ising universality class (i.e. both the standard ones $y = 1$ and $\frac{15}{8}$ and the 'borderline' ones $y = \frac{13}{24}$, $\frac{11}{8}$ and $\frac{5}{3}$) become fractal dimensions of suitable sets of links (or sites) of the Ising percolation model at the critical point. In particular, $y = \frac{15}{8}$ is the fractal dimension of the percolating cluster, $y = 1$ is related to the correlation length, $y = \frac{13}{24}$ is the fractal dimension of the red bonds (see [59]), $y = \frac{5}{3}$ is the fractal dimension of the percolating cluster in the presence of a boundary (see [60]) and $y = \frac{11}{8}$ is the fractal dimension of the hull (see [61]). Unfortunately, the operator in which we are interested, being irrelevant, cannot be realized as a fractal dimension, but the coincidence of the other indices strongly supports the idea that it should also appear as the subleading dimension of some suitably chosen set of links.

Some theoretical justification of this remarkable coincidence of critical indices and fractal dimensions can be found in an interesting conjecture that was proposed for the first time in [62] and then discussed in detail in [59, 60]. According to this conjecture, Ising percolation is described by the $q \to 1$ limit of the tricritical q-state Potts model in exactly the same way in which the $q \to 1$ limit of ordinary q-state Potts describes standard percolation. The operator content of the $q \to 1$ limit of the tricritical q-state Potts model can be studied with the same Coulomb gas techniques discussed above. It turns out that it contains (together with other operators) the non-unitary extension of the Ising model, and thus explains the above coincidence of critical indices and fractal dimensions. Note that this conjecture is further supported by the identification as fractal dimensions of suitable sets of links of other critical indices that belong to the $q \to 1$ limit of the tricritical Potts but that are outside the non-unitary Ising class—see [60] for a discussion.

Appendix B. Resummation of simple test functions

In this appendix we consider a simple test function which behaves as (8) and whose perturbative expansion around $g = 0$ is divergent but Borel summable. We show that many terms are needed in order to obtain the correct results, and, even worse, that in this case the standard method to set the error bars does not work properly. The estimated errors are much smaller than the difference between the estimate and the exact value.

Consider the function

$$Z(g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}x \exp\left(-\frac{1}{2}x^2 - \frac{1}{4!}gx^4\right). \tag{B1}$$

Its expansion in powers of g, $Z(g) = \sum_k Z_k g^k$, is Borel summable, and the large-order behaviour of the kth-order coefficient Z_k is given by

$$Z_k = (-1)^k \frac{(4k-1)!!}{4!^k k!} \propto \left(-\tfrac{2}{3}\right)^k (k-1)!\,[1 + \mathrm{O}(1/k)]. \tag{B2}$$

The function $Z(g)$ is analytic in the complex plane with a cut along the negative real axis, and, in particular, it is analytic for $g = 1$. For $\delta \equiv 1 - g \to 0$ it behaves as

$$Z(g) = Z_0 + Z_1\delta + \mathrm{O}(\delta^2) \tag{B3}$$

where $Z_0 = 0.918\,9189\ldots$ and $Z_1 = -0.057\,3155\ldots$. In this case, in which the function is analytic, the resummation method we presented in the introduction provides good estimates of the constants appearing in (B3). One indeed obtains $Z_0 = 0.9189(1)$ and $Z_1 = -0.0572(3)$ from the fifth-order series, and $Z_0 = 0.918\,919(1)$ and $Z_1 = -0.057\,315(3)$ from the 10th-order series†. Most importantly, the method provides correct estimates of the errors.

In order to reproduce a non-analytic behaviour similar to (21), we consider the function

$$B(b, g) = Z(g) + c(1 - g)^{1+b}. \tag{B4}$$

Setting $c = Z_1$, we have for $g \to 1$

$$B(b, g) = Z_0 + Z_1\delta\left(1 + \delta^b\right) + \mathrm{O}(\delta^2). \tag{B5}$$

We apply the same resummation procedure used for $Z(g)$ to the perturbative expansion of $B(b, g)$. To reproduce the correction predicted in the Ising case, we fix $b = \frac{1}{7}$. The results of the analysis are now much less satisfactory. Indeed, we find $Z_0 = 0.916(6)$ and $Z_1 = -0.112(2)$ from the fifth-order series, and $Z_0 = 0.918(1)$ and $Z_1 = -0.103(6)$ from the 10th-order series. The estimate of Z_0 is not as precise as before, but the error is still correct. This is not surprising since the non-analyticity is rather weak here, the non-analytic corrections being of the order of δ^{1+b}. On the other hand, the estimate of Z_1, which is determined by resumming the $\mathrm{d}B(b, g)/\mathrm{d}g$ (here, the non-analytic corrections are stronger, of the order of δ^b), is very imprecise and the estimate of the error, which is obtained from the stability analysis, is completely incorrect: the five-loop estimate differs from the exact value by more than 25 estimated error bars! Moreover, extending the series appears to be of little help. We conjecture that a similar phenomenon is happening in the FT estimates for $N = 1$. Although the perturbative results indicate $\omega_1 \approx \frac{4}{3}$ with a tiny error, the correct result is sensibly different.

We have also considered the case in which we add a term of the form $Z_1 g/\log(1 - g)$, which mimicks the behaviour of the β-function for $N \geqslant 3$, observing completely analogous deviations.

We have repeated the exercise by considering a non-analytic singularity similar to that expected in three dimensions, i.e. by setting $b \simeq 1$. For example, for $b = \frac{9}{10}$ we find $Z_0 = 0.917(1)$ and $Z_1 = -0.068(4)$ from the fifth-order series, $Z_0 = 0.9186(1)$ and $Z_1 = -0.060(2)$ from the 10th-order series. As expected, the effect of the non-analyticity is much smaller and the errors are reasonable, although slightly underestimated.

Appendix C. Asymptotic expansion of large-N integrals

In this appendix we wish to compute the asymptotic expansion for $\Theta \to 0$ of integrals of the form

$$I_n(f, \Theta) = \int_0^\infty \mathrm{d}u\, \frac{f(u)}{[\Theta + \delta(u)]^n} \tag{C1}$$

† The estimates and their errors are obtained using the procedure of [12]. The estimate is obtained from the 'optimal' values of the two free parameters introduced in the procedure (b and α), which are determined by maximizing the stability of the results with respect to the order of the series analysed. The errors are related to the stability of the results with respect to variations of the free parameters b and α around their optimal values.

where $f(u) \sim u^{-p}$ for $u \to \infty$, and n and p are integers satisfying $n \geqslant 1$, $p \geqslant 2$. The function $\delta(u)$ is given by

$$\delta(u) = -\frac{2}{u\xi} \log \frac{1-\xi}{1+\xi} \tag{C2}$$

where

$$\xi(u) = \sqrt{\frac{u}{u+4}}. \tag{C3}$$

The results presented here extend appendix A of [12] to two dimensions. We wish to compute the leading non-analytic contributions to the asymptotic expansion. For this purpose, we can replace $\delta(u)$ and $f(u)$ with their leading behaviour for $u \to \infty$ and write

$$I_n(f, \Theta) \approx \int_{1/\Lambda}^{\infty} \frac{\mathrm{d}u\, u^{-p}}{[\Theta + (2\log u)/u]^n} \tag{C4}$$

where Λ is an arbitrary cut-off satisfying $0 < \Lambda < 1$. Then we make the substitution

$$\frac{2}{u} \log u = y. \tag{C5}$$

For $y \to 0$, $u \to \infty$, equation (C5) can be solved, obtaining the asymptotic expansion

$$\frac{1}{u} = -\frac{y}{2\log(y/2)} \left\{ 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{n} a_{nm} \frac{[\log(-\log(y/2))]^m}{\log^n(y/2)} \right\}. \tag{C6}$$

The first coefficients are $a_{11} = a_{22} = 1$ and $a_{21} = -1$.

Substituting this expression in (C4) and keeping only the leading contributions, we obtain

$$I_n(f, \Theta) \approx \int_0^{\Lambda} \frac{\mathrm{d}y}{y u^{p-1}} \frac{1}{(\Theta + y)^n} \tag{C7}$$

where analytic terms have been systematically neglected.

Since $p \geqslant 2$, we see that $I_n(f, \Theta)$ can be written as a sum of terms of the form

$$K_{nmp}(\Theta) = \int_0^{\Lambda} \mathrm{d}y\, \frac{[\log(-\log y)]^p}{(-\log y)^m (\Theta + y)^n} \tag{C8}$$

with m, n and p being integers. The non-analytic terms are due to the integrals with $n \geqslant 1$, and thus we consider only this case. Now, observe that we need to consider $n = 1$ only, since

$$K_{nmp}(\Theta) = \frac{(-1)^{n-1}}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}\Theta^{n-1}} K_{1mp}(\Theta). \tag{C9}$$

Then, also note that

$$[\log(-\log y)]^p = \lim_{\epsilon \to 0} \left[\frac{(-\log y)^{\epsilon} - 1}{\epsilon} \right]^p. \tag{C10}$$

Thus, it is enough to consider $K_{1\alpha 0}$, where α is a real number. In the following we assume $\alpha > 1$. The final result, however, will be correct for all values of α. To compute the asymptotic expansion, first perform a Mellin transformation, rewriting

$$K_{1\alpha 0} = -\int_{-1/2-\mathrm{i}\infty}^{-1/2+\mathrm{i}\infty} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \frac{\pi}{\sin \pi s} \Theta^s R_\alpha(\Lambda, s) \tag{C11}$$

where

$$R_\alpha(\Lambda, s) = \int_0^\Lambda \frac{\mathrm{d}y}{(-\log y)^\alpha} y^{-1-s} = \int_{-\log\Lambda}^\infty \frac{\mathrm{d}t}{t^\alpha} \mathrm{e}^{st}. \tag{C12}$$

The previous equation defines $R_\alpha(\Lambda, s)$ for $\mathrm{Re}\, s \leqslant 0$. By rotating the t contour one can obtain an analytic continuation in the domain $\mathrm{Re}\, s > 0$ with a cut along the positive real axis. In the following, we need the discontinuity at the cut. A simple calculation gives

$$R_\alpha(\Lambda, s+) - R_\alpha(\Lambda, s-) = \int_C \frac{\mathrm{d}t}{t^\alpha} \mathrm{e}^{st} = \frac{2\pi \mathrm{i}}{\Gamma(\alpha)} s^{\alpha-1} \tag{C13}$$

where C is a contour running counterclockwise around the negative t-axis. We also need $R_\alpha(\Lambda, 0) = (-\log\Lambda)^{1-\alpha}/(\alpha - 1)$. In order to compute the asymptotic expansion of $K_{1\alpha 0}(\Theta)$, deform the s-integral, so that it goes around the positive s-axis. Taking into account the pole at $s = 0$ we obtain

$$\begin{aligned} K_{1\alpha 0}(\Theta) = R_\alpha(\Lambda, 0) &- \int_0^\mu \frac{\mathrm{d}s}{2\pi \mathrm{i}} \frac{\pi}{\sin \pi s} \Theta^s [R_\alpha(\Lambda, s+) - R_\alpha(\Lambda, s-)] \\ &- \int_{C_+ + C_-} \frac{\mathrm{d}s}{2\pi \mathrm{i}} \frac{\pi}{\sin \pi s} \Theta^s R_\alpha(\Lambda, s) \end{aligned} \tag{C14}$$

where $0 < \mu < 1$ is arbitrary and $C_\pm = \{s\colon\ \mathrm{Re}\, s = \mu \quad \pm \mathrm{Im}\, s > 0\}$. The integral over the lines $C_\pm$ is of the order of Θ^μ and can therefore be discarded. In order to compute the integral over the cut, we make the substitution $-s \log\Theta = t$, expand the integrand in powers of $1/\log\Theta$ and replace the upper integration limit $-\mu\log\Theta$ with ∞—again we make an error of the order of Θ^μ. The final integrations are trivial. We obtain finally

$$K_{1\alpha 0}(\Theta) \approx \frac{1}{\alpha - 1} (-\log\Lambda)^{1-\alpha} - (-\log\Theta)^{1-\alpha} \sum_{k=0}^\infty b_k \frac{\Gamma(2k + \alpha - 1)}{\Gamma(\alpha)} \left(\frac{\pi}{\log\Theta}\right)^{2k} \tag{C15}$$

where the coefficients b_k are defined by

$$\frac{1}{\sin x} = \sum_{k=0}^\infty b_k x^{2k-1}. \tag{C16}$$

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